International Journal of Theoretical Physics, Vol. 7, No. 1 (1973), pp. 39–47

Galilean Invariance and Magnetic Monopoles

H. BACRY†

Université d'Aix-Marseille

and J. KUBAR-ANDRE†

Centre de Physique Théorique, CNRS, Marseille

Received: 22 February 1972

Abstract

The existence of Dirac monopoles is shown to be incompatible with Galilean invariance. A discussion follows on the interpretation of monopoles physics in a Galilean approximation.

I

Some authors (Kerner, 1970; Schwinger, 1969; Zwanziger, 1968) have recently investigated the physical implications of the existence of particles endowed with *both* electric and magnetic charge. In particular, Schwinger has suggested that quarks could be such particles (dyons); such a property would give a very nice explanation of some fundamental properties in relation with strong interactions.

Existence of monopoles or dyons can be admitted in a very natural way in the framework of special relativity. The theory is usually written in a manifestly covariant way. At first sight, it seems natural and simpler to look for a Galilean interpretation of such particles. Unfortunately, such an interpretation is not obvious because the beautiful symmetry between electric and magnetic fields in Maxwell equations is no longer true in Galilean invariant electrodynamics.

One of the assumptions usually made concerns the electromagnetic force acting on a particle of electric charge e and magnetic charge g.

$$\mathbf{f} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + g(\mathbf{B} - \mathbf{v} \times \mathbf{E}) \tag{1}$$

† Postal Address: Centre de Physique Théorique—CNRS—31, chemin Joseph Aiguier—13 Marseille 9e (France).

Such a formula is obviously induced by relativity where a boost of speed $\mathbf{v}$ is known to transform the electric and magnetic vectors in the following way

$$\mathbf{E}' = \mathbf{E} + \gamma \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{\gamma^2}{1+\gamma}\frac{\mathbf{v}}{c} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right)$$

$$\mathbf{B}' = \mathbf{B} - \gamma \frac{\mathbf{v}}{c} \times \mathbf{E} - \frac{\gamma^2}{1+\gamma}\frac{\mathbf{v}}{c} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

where $\gamma = [1 - (v^2/c^2)]^{-1/2}$.

It is evident that, by neglecting terms of order two in v/c, one gets

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}$$

$$\mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}$$

Such a way of getting the non-relativistic limit is in fact careless; this is not surprising due to the fact that c is taken as the unit speed instead of making c going to infinity in adequate formulae.

Before giving a logical description of a Galilean invariant theory, we intend to present a simple argument against formula (1). In the proper Galilean frame, where the speed of the particle is zero at a given time, one would require

$$\mathbf{f} = e\mathbf{E} + g\mathbf{B}$$

Suppose the field transforms under a boost in the following way

$$\mathbf{E} \to \mathbf{E}' = \mathbf{E} + \alpha\mathbf{v} \times \mathbf{B}$$

$$\mathbf{B} \to \mathbf{B}' = \mathbf{B} - \lambda\mathbf{v} \times \mathbf{E}$$

where α and λ are constants.

Make a boost $\mathbf{v}$, then a boost $\mathbf{v}'$. One gets

$$\mathbf{f} = e[\mathbf{E} + \alpha(\mathbf{v} + \mathbf{v}') \times \mathbf{B}] + g[\mathbf{B} - \lambda(\mathbf{v} + \mathbf{v}') \times \mathbf{E}]$$
$$- \alpha\lambda[e\mathbf{v}' \times (\mathbf{v} \times \mathbf{E}) + g\mathbf{v}' \times (\mathbf{v} \times \mathbf{B})]$$

Galilean law of additivity of speeds is only satisfied if $\alpha\lambda = 0$, that is if α or λ is zero. The correct formulae are, in fact,

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

$$\mathbf{B}' = \mathbf{B} \qquad (2)$$

They are the formulae implicitly used by physicists before Maxwell introduced the 'displacement' term proportional to $\partial\mathbf{E}/\partial t$ in his equations. In fact, equations (2) have been rigorously derived by Levy-Leblond (1965, 1967) from Galilean invariance.

II

The Maxwell equations involving monopoles are

$$\begin{aligned} \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j}_e \\ \nabla . \mathbf{E} &= \rho_e \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= -\mathbf{j}_m \\ \nabla . \mathbf{B} &= -\mathbf{j}_m \end{aligned} \tag{3}$$

or, covariantly,

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j_e{}^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= j_m{}^\nu \end{aligned}$$

where the four-vectors $j_e{}^\nu = (\rho_e, \mathbf{j}_e)$ and $j_m{}^\nu = (\rho_m, \mathbf{j}_m)$ denote the electric current and magnetic current, respectively. The generalized Maxwell equations imply both conservation of electric and magnetic charge. The velocity of light c is made explicit in view of the Galilean limit $c \to \infty$.

In a given external electromagnetic field, the motion of a particle of mass m_1, electric charge e_1 and magnetic charge g_1 is described by the following Lagrangian

$$L = -m_1 c^2 \sqrt{(1 - v_1{}^2/c^2)} + e_1 \mathbf{A}.\mathbf{v}_1 - e_1 \Phi + \frac{1}{c^2} g_1 \mathbf{A}^*.\mathbf{v}_1 - g_1 \Phi^* \tag{4}$$

where the four-potentials

$$\begin{aligned} A^\mu &= (\Phi, \mathbf{A}) \\ A^{*\mu} &= \left(\Phi^*, \frac{1}{c^2}\mathbf{A}^*\right) \end{aligned}$$

are solutions of the following equations in space-time domains free of charge

$$\begin{aligned} F_{\mu\nu} &= \partial_\nu A_\mu - \partial_\mu A_\nu \\ \tilde{F}_{\mu\nu} &= \partial_\nu A_\mu{}^* - \partial_\mu A_\nu{}^* \end{aligned}$$

or, in a non-covariant way,

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad & \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\nabla \times \mathbf{A}^*, \qquad & \mathbf{B} &= -\nabla\Phi^* - \frac{1}{c^2}\frac{\partial \mathbf{A}^*}{\partial t} \end{aligned} \tag{5}$$

Now, suppose the electromagnetic field to be produced by a single particle (m_2, e_2, g_2). As long as m_2 is large compared to m_1, the particle 2 can be considered as fixed and the potentials and fields are given by

$$\Phi = \frac{e_2}{r} \qquad \mathbf{A} = g_2 \frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{r}.\mathbf{n})}$$

$$\Phi^* = \frac{g_2}{r} \qquad \mathbf{A}^* = -e_2 \frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{r}.\mathbf{n})}$$

$$\mathbf{E} = e_2 \frac{r}{r^3}$$

$$\mathbf{B} = g_2 \frac{\mathbf{r}}{r^3}$$

The choice of potentials corresponds to the Dirac choice of semi-infinite singularity or string (Dirac, 1948) (the unit vector **n** is arbitrary).

III

In Galilean electrodynamics, the field equations are easily derived from equations (3) by making c go to infinity. One gets

$$\begin{aligned} \nabla \times \mathbf{B} &= \mathbf{j}_e \\ \nabla . \mathbf{E} &= \rho_e \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= -\mathbf{j}_m \\ \nabla . \mathbf{B} &= \rho_m \end{aligned} \tag{6}$$

Except for source terms, these equations have been derived by Levy-Leblond (1965, 1967) as Galilean invariant equations for particles of zero mass and helicity one. The only difference between (3) and (6) lies in the absence of the Maxwell term $\partial \mathbf{E}/\partial t$. This has important physical implications. First, we note that instead of $\operatorname{div} \mathbf{j}_e + (\partial \rho_e/\partial t) = 0$, one has

$$\operatorname{div} \mathbf{j}_e = 0$$

One can choose between two attitudes:

(i) One requires that $\mathbf{j}_e$ is still an electric current, that is to say it characterizes charges in motion. The conservation of electric charge can only be an assumption since it is not a consequence of field equations. If we do so, we get the condition $\partial \rho_e/\partial t = 0$ which implies constant densities. *In such a scheme, electrically charged particles cannot exist:* they would obviously violate the law $\partial \rho_e/\partial t = 0$. Only electrically charged fluids are compatible with this interpretation (for instance permanent electric currents in wires).

(ii) One would better choose the following alternative: the vector density $\mathbf{j}_e$ is a source for magnetic fields, *a source which has nothing to do with electric charges.* In such a theory, an electric charge in motion cannot produce any magnetic field.

Because we are concerned with particles, it is more natural to adopt the last attitude. Note that Galilean relativity implies conservation of magnetic charge, a fact which creates a deep dissymmetry between electric and magnetic effects. Such a dissymmetry also appears in the non-relativistic limit of the Lagrangian (Inonu & Wigner, 1953). One gets

$$L = \tfrac{1}{2}m_1 v_1^2 + e_1 \mathbf{A}_1 . \mathbf{v}_1 - e_1 \Phi - g_1 \Phi^* \tag{7}$$

Equations (5) become

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad & \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\nabla \times \mathbf{A}^* & \mathbf{B} &= -\nabla\Phi^* \end{aligned} \tag{8}$$

It is easy to check Galilean invariance of the Lagrangian (7) by performing a pure Galilean transformation of speed $\mathbf{u}$. Potentials and fields transform as follows:

$$\begin{aligned} \Phi' &= \Phi - \mathbf{u}.\mathbf{A} \qquad & \mathbf{A}' &= \mathbf{A} \\ \Phi'^* &= \Phi^* & \mathbf{A}^{*\prime} &= \mathbf{A}^* - \mathbf{u}\Phi^* \end{aligned} \tag{9}$$

$$\begin{aligned} \mathbf{E}' &= \mathbf{E} + \mathbf{u} \times \mathbf{B} \\ \mathbf{B}' &= \mathbf{B} \end{aligned} \tag{2}$$

Now suppose that particle 2 which produces the external field has a speed $\mathbf{v}_2$. In its rest system at time t, one has

$$\begin{aligned} \Phi' &= \frac{e_2}{r} \qquad & \mathbf{A}' &= g_2 \frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{r}.\mathbf{n})} \\ \Phi^{*\prime} &= \frac{g_2}{r} & \mathbf{A}^{*\prime} &= -e_2 \frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{r}.\mathbf{n})} \end{aligned}$$

$$\mathbf{E}' = e_2 \frac{\mathbf{r}}{r^3}$$

$$\mathbf{B}' = g_2 \frac{\mathbf{r}}{r^3}$$

Transformation formulae (9) and (2) provide us with potentials and fields measured in the laboratory

$$\begin{aligned} \Phi &= \frac{e_2}{r} + g_2 \mathbf{v}_2 \frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{r}.\mathbf{n})} \qquad & \mathbf{A} &= g_2 \frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{r}.\mathbf{n})} \\ \Phi^* &= \frac{g_2}{r} & \mathbf{A}^* &= -e_2 \frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{r}.\mathbf{n})} + g_2 \frac{\mathbf{v}_2}{r} \end{aligned} \tag{10}$$

$$\mathbf{E} = e_2 \frac{\mathbf{r}}{r^3} - g_2 \frac{\mathbf{v}_2 \times \mathbf{r}}{r^3}$$
$$\mathbf{B} = g^2 \frac{\mathbf{r}}{r^3} \tag{11}$$

This last equation gives the proof that an electric charge in motion cannot produce a magnetic field. On the other hand, a magnetic charge in motion does produce an electric field.

It is interesting to look for the Lagrangian describing the motion of particle 1 in the field produced by particle 2. This is readily obtained by bringing the expressions (10) into the Lagrangian (7). One obtains

$$L = \tfrac{1}{2} m_1 v_1{}^2 - \frac{e_1 e_2 + g_1 g_2}{r} + e_1 g_2 \frac{(\mathbf{v}_1 - \mathbf{v}_2).(\mathbf{r} \times \mathbf{n})}{r(r - \mathbf{r}.\mathbf{n})}$$

and the Lagrange equation is

$$m_1 \frac{d\mathbf{v}_1}{dt} = (e_1 e_2 + g_1 g_2) \frac{\mathbf{r}}{r^3} + e_1 g_2 \frac{(\mathbf{v}_1 - \mathbf{v}_2) \times \mathbf{r}}{r^3} \tag{12}$$

an equation which could be derived directly from Newton's law and equations (11).

By a symmetrical argument, one would readily obtain the equation of motion of particle 2 in the field produced by particle 1 (one must replace 1 by 2 and $\mathbf{r}$ by $-\mathbf{r}$)

$$m_2 \frac{dv_2}{dt} = -(e_1 e_2 + g_1 g_2) \frac{\mathbf{r}}{r^3} - e_2 g_1 \frac{(\mathbf{v}_2 - \mathbf{v}_1) \times \mathbf{r}}{r^3} \tag{13}$$

If one requires the validity of Newton's third law (or, equivalently, translational invariance), one must write

$$m_1 \frac{d\mathbf{v}_1}{dt} + m_2 \frac{d\mathbf{v}_2}{dt} = 0$$

that is

$$e_1 g_2 + e_2 g_1 = 0$$

This condition is very strong. Consider three particles endowed with non-zero electric charges. One gets

$$e_1 g_2 + e_2 g_1 = e_2 g_3 + e_3 g_2 = e_3 g_1 + e_1 g_3 = 0$$

The condition $e_1 e_2 e_3 \neq 0$ requires one to write successively

$$g_2 = -\frac{e_2}{e_1} g_1 \qquad g_3 = -\frac{e_3}{e_1} g_1 \qquad \frac{e_2 e_3}{e_1} g_1 = 0$$

which leads to $g_1 = 0$. Therefore, *there is no place for mixed charged particles*. Moreover, if one writes equation (7) for a system composed of one electric charge e_1 and one magnetic monopole g_2, one would get

$$e_1 g_2 = 0$$

which needs to reject the existence of magnetic charge *or* electric charge.

IV

As shown by Zwangiger and Schwinger, the correct (therefore, relativistic) theory of two interacting particles must involve the two quantities $e_1 e_2 + g_1 g_2$ and $e_1 g_2 - e_2 g_1$. The presence of the last quantity instead of $e_1 g_2$ guarantees the validity of Newton's third law. If we make the substitution $e_1 g_2 \to (e_1 g_2 - e_1 g_1)$ in equation (12) and $e_2 g_1 \to (e_2 g_1 - e_1 g_2)$ in equation (13), we are led to two equations of motion *in agreement with* Galilean invariance but without intermediate field. It is a theory with *direct interactions*. We arrive at the conclusion that a *Galilean invariant theory of magnetic charges is incompatible with the existence of the electromagnetic field carrying electromagnetic interactions*. According to the conclusion of Section III, we are left with the following possibilities in a Galilean theory

(i) Existence of electric and magnetic monopoles without field.
(ii) Existence of an electromagnetic field and electric charges alone (*or* magnetic charges alone).

However, the Galilei group suffers two interpretations: either it is considered as *the* kinematical group of our universe or as the approximation of some other kinematical groups when some restrictions are fulfilled. Until now, we have examined the first point of view. From the second one, it is known (Inonu & Wigner, 1953) that the Galilei group is obtained, through a *speed-space contraction* (Bacry & Lévy-Leblond, 1968), from the Poincaré group; this means that it is a good kinematical group whenever speeds are small compared to *c and* space intervals small compared to time ones (Levy-Leblond, 1965). It is necessary to examine monopoles from this point of view.

First, we suppose magnetic monopoles not to exist. We make $c = 1$ which implies that electric and magnetic fields are measured with the same unit. If speeds are small (say $v \sim \varepsilon$) the Lorentz force is of the order $E + \varepsilon B$. The force must be weak enough in order to avoid speeds increasing too much. This condition implies E to be of the order of εB, that is *electric fields must be small compared to magnetic fields*. This condition would have to be added to that on speeds and space intervals if we want to use Galilean electrodynamics.† By neglecting terms of order ε^2, it is readily seen that the Galilean limit for boost formulae is

$$\mathbf{E} \to \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

$$\mathbf{B} \to \mathbf{B}$$

This has already been obtained in the context of an exact symmetry.

Similar arguments could be used in a situation where only magnetic monopoles are supposed to exist. Besides the conditions on speeds and space intervals, one would have to suppose small magnetic fields and the following formulae

$$\mathbf{E} \to \mathbf{E}$$

$$\mathbf{B} \to \mathbf{B} - \mathbf{v} \times \mathbf{E}$$

and the Lorentz force would read‡

$$\mathbf{f} = g(\mathbf{B} - \mathbf{v} \times \mathbf{E})$$

Obviously, the interesting question concerns the case where both electric and magnetic charges are present in Nature. The answer must take into account the Dirac derivation of charge quantization. It is well known, following Dirac's results, that electric charges are much smaller than magnetic ones, the ratio $\varepsilon = eg^{-1}$ is of the order $\frac{1}{137}$. It can be readily seen that the first-order Lorentz force

$$\mathbf{f} = e\mathbf{E} + g(\mathbf{B} - \mathbf{v} \times \mathbf{E}) \tag{14}$$

is a good Galilean approximation when ef^{-1}, BE^{-1} and v are of the order of ε. Nevertheless, this formula corresponds to a rather strange situation where large magnetic charges and weak magnetic fields are simultaneously involved. An assumption of the type $EB^{-1} \sim \varepsilon$ would lead to too strong forces, a fact which cannot be admitted in the framework of a Galilean theory since it is in contradiction with our assumption of first-order forces. Obviously, we could assume EB^{-1}, eg^{-1} and v of different orders in ε in order to make formula (14) valid in the Galilean limit, but it does not seem to us that such an approach could be interesting. We only intended to underline the difficulties which appear in the elaboration of a Galilean consistent theory of magnetic monopoles, the main important fact being the need of direct interactions without field support.

† This condition could be obtained through the contraction process by using, instead of the Poincaré group itself, another inhomogenization of the Lorentz group involving 'field translations'.

‡ In fact, when only one kind of monopoles is present in Nature, it is absolutely impossible to distinguish between electric and magnetic ones, by using electromagnetic interactions.

Acknowledgement

The authors are grateful to A. Grossmann for fruitful discussions and to S. Glashow for his critical comments of the manuscript.

References

Bacry, H. and Levy-Leblond, J. M. (1968). *Journal of Mathematical Physics*, **9**, 1605.
Dirac, P. A. M. (1948). *Physical Review*, **74**, 817.
Inonu, E. P. and Wigner, E. (1953). *Proceedings of the National Academy of Sciences, U.S.A.* **39**, 510.
Kerner, E. H. (1970). *Journal of Mathematical Physics*, **11**, 39.
Levy-Leblond, J. M. (1965). These de Doctorat—Orsay.
Levy-Leblond, J. M. (1967). *Communications in Mathematical Physics*, **6**, 286.
Schwinger, J. (1969). *Science, N.Y.* **165**, 757.
Zwanziger, D. (1968). *Physical Review*, **176**, 1480.